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1978 J. Phys. A: Math. Gen. 11 697

(http://iopscience.iop.org/0305-4470/11/4/010)

J. Phys. A: Math. Gen., Vol. 11, No. 4, 1978. Printed in Great Britain. © 1978

A general set of relations involving 3-j symbols

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Received 14 October 1977

Abstract. It is shown that the sums

$$\sum_{l'=0}^{l}\left(\prod_{i=1}^{n}\frac{1}{(2l'+2k_i+1)}\right)\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2$$

involving squares of particular 3-j symbols, can be expressed in terms of gamma functions and higher transcendentals, for complex k_i and for any natural number n. Simplified formulae are derived for integral and half-integral k_i values.

1. Introduction

In investigating recursion relations for the exact solution of the non-relativistic helium atom problem, Morgan (1975) showed with the aid of standard recursion techniques the following relations involving squares of 3-j symbols:

$$S_l=\sum_{l'=0}^{l}\frac{1}{2l'-1}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2=-\delta_{l0} \qquad (\forall l\in\mathbb{N}). \tag{1.1}$$

It was conjectured by Morgan (1975) and proved by the present authors (Vanden Berghe and De Meyer 1976) that:

$$\bar{S}_l=\sum_{l'=0}^{l}\frac{1}{2l'+3}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2-\sum_{l'=0}^{l}\frac{1}{2l'+1}\begin{pmatrix} l & l'+1 & l-l'+1 \\ 0 & 0 & 0 \end{pmatrix}^2=0 \qquad (\forall l\in\mathbb{N}). \tag{1.2}$$

Using certain properties of hypergeometric functions, an alternative though not completely justified proof of (1.2) was given by Rashid (1976). Recently it was demonstrated by Morgan (1976) that formulae (1.1) and (1.2) are only two special cases of a more general result:

$$S_{l,J}(z)=\sum_{l'=0}^{l}\left(\frac{z(2l+z+1)}{(2l+z)(z-1)}\frac{1}{2l'+z+1}-\frac{1}{2l'+z-1}\right)\begin{pmatrix} l & l'+J & l-l'+J \\ 0 & 0 & 0 \end{pmatrix}^2=0$$
$$(\forall l\in\mathbb{N}), \tag{1.3}$$

which is valid for integral J and for all real or complex $z\neq 0$, such that the sum makes

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sense. For $z=0$, the relation (1.3) has to be replaced by

$$\sum_{l'=0}^{l}\frac{1}{2l'-1}\begin{pmatrix}l & l'+J & l-l'+J\\ 0 & 0 & 0\end{pmatrix}^2=-\frac{1}{2J+1}\delta_{l0}.$$

A second derivation of (1.3), based on the Neumann expansions of r_{12} and $1/r_{12}$, where r_{12} is the distance between two vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ was also given by Morgan (1977).

In the present paper, a method is outlined to evaluate the quantities $S_l^{(n)}(k_1, k_2, \ldots, k_n)$ defined by:

$$S_l^{(n)}(k_1,k_2,\ldots,k_n)=\sum_{l'=0}^{l}\left(\prod_{i=1}^{n}\frac{1}{2l'+2k_i+1}\right)\begin{pmatrix}l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2, \tag{1.4}$$

for arbitrary non-negative integers l and n, and for real k_i ($i=1,2,\ldots$). The particular and slightly more difficult case whereby two or more of the k_i values coincide, has to be treated separately. We also give reduced and simplified formulae for integral and half-integral k_i values.

2. Evaluation of $S_l^{(1)}(k)$

In deriving an expression for $S_l^{(1)}(k)$ defined by (1.4), use will be made of the result (1.1) and of the formula:

$$\sum_{l'=0}^{l}\frac{1}{(l-l'+1)(2l'-1)}\begin{pmatrix}l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2=\frac{1}{(l+1)(2l+1)}-2\delta_{l0}\qquad(\forall l\in\mathbb{N}), \tag{2.1}$$

which is immediately obtained from formula (10) of Morgan (1975), by changing l' to $l-l'$ and by taking into account the relation

$$\begin{pmatrix}l & l'+J & l-l'+J\\ 0 & 0 & 0\end{pmatrix}^2=\frac{(2J)!(l+J)!^2(2l+1)!}{J!^2l!^2(2l+2J+1)!}\begin{pmatrix}l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2. \tag{2.2}$$

This last relation follows directly from the definition of the occurring 3-j symbol.

With the aid of (1.1) and of the identity

$$\frac{1}{(l-l'+1)(2l'-1)}=\frac{1}{2l+1}\left(\frac{1}{l-l'+1}+\frac{2}{2l'-1}\right)$$

one finds that

$$\frac{1}{2l+1}\sum_{l'=0}^{l}\frac{1}{l-l'+1}\begin{pmatrix}l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2=\frac{1}{(l+1)(2l+1)}-2\delta_{l0}+\frac{2}{2l+1}\delta_{l0},$$

or

$$\sum_{l'=0}^{l}\frac{1}{l-l'+1}\begin{pmatrix}l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2\equiv\sum_{l'=0}^{l}\frac{1}{l'+1}\begin{pmatrix}l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2=\frac{1}{l+1}. \tag{2.3}$$

Another important identity is the following:

$$\begin{pmatrix}l & l'-1 & l-l'+1\\ 0 & 0 & 0\end{pmatrix}^2=\frac{l'(2l-2l'+1)}{(2l'-1)(l-l'+1)}\begin{pmatrix}l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2. \tag{2.4}$$

We next define quantities $A_l(k)$ for positive integral values of k and non-negative integers l, by

$$A_l(k)=\sum_{l'=0}^{l}\frac{1}{2l'+2k+1}\begin{pmatrix} l & l'+k-1 & l-l'+k-1\\ 0 & 0 & 0\end{pmatrix}^2. \tag{2.5}$$

The $A_l(k)$ are consecutively reduced for $l\neq 0$ to

$$\begin{aligned}A_l(k)=&\frac{1}{2l+2k+1}\begin{pmatrix} l & l+k-1 & k-1\\ 0 & 0 & 0\end{pmatrix}^2\\ &+\sum_{l'=0}^{l-1}\frac{1}{2l'+2k+1}\begin{pmatrix} l & l'+k-1 & l-l'+k-1\\ 0 & 0 & 0\end{pmatrix}^2\\ &=\frac{(2k-2)!(2l)!(l+k-1)!^2}{(k-1)!^2l!^2(2l+2k-1)!}\Bigg[\frac{1}{2l+2k+1}\\ &+(2l+1)\sum_{l'=0}^{l}\frac{l'(2l-2l'+1)}{(2l'+2k-1)(l-l'+1)(2l'-1)}\begin{pmatrix} l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2\Bigg]\end{aligned}$$

by first separating the term with $l'=l$, then changing l' to $l'-1$ in the remaining sum, and using (2.2) and (2.4). One can easily check that

$$\begin{aligned}&\frac{l'(2l-2l'+1)}{(2l'+2k-1)(l-l'+1)(2l'-1)}\\ &\quad=\frac{1}{2l+2k+1}\left(\frac{(k+l)(2k-1)}{k}\frac{1}{2l'+2k+1}+\frac{2k+l}{k}\frac{1}{2l'-1}\right.\\ &\quad\left.-(l+1)\frac{1}{(l-l'+1)(2l'-1)}\right).\end{aligned}$$

Therefore, and with the help of (1.1) and (2.1), $A_l(k)$ $(l\neq 0)$ can be transformed to

$$A_l(k)=\frac{(2k-2)!(2l+1)!(l+k-1)!^2(k+l)(2k-1)}{(k-1)!^2l!^2(2l+2k-1)!k(2l+2k+1)}\sum_{l'=0}^{l}\frac{1}{2l'+2k-1}\begin{pmatrix} l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2,$$

or

$$\begin{aligned}A_l(k)\equiv&\sum_{l'=0}^{l}\frac{1}{2l'+2k+1}\begin{pmatrix} l & l'+k-1 & l-l'+k-1\\ 0 & 0 & 0\end{pmatrix}^2\\ &=\sum_{l'=0}^{l}\frac{1}{2l'+2k-1}\begin{pmatrix} l & l'+k & l-l'+k\\ 0 & 0 & 0\end{pmatrix}^2,\end{aligned} \tag{2.6}$$

where in the last step (2.2) has been used in reversed order. From (2.6) one finds, by making use of (2.2) twice, that

$$\begin{aligned}&\frac{(2k-2)!(l+k-1)!^2}{(k-1)!^2(2l+2k-1)!}\sum_{l'=0}^{l}\frac{1}{2l'+2k+1}\begin{pmatrix} l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2\\ &\quad=\frac{(2k)!(l+k)!^2}{k!^2(2l+2k+1)!}\sum_{l'=0}^{l}\frac{1}{2l'+2k-1}\begin{pmatrix} l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2.\end{aligned} \tag{2.7}$$

It is seen by direct calculation that (2.7) is also valid for $l=0$. Furthermore it is easy to demonstrate that (1.3) with $J=0$ is a consequence of (2.7) if z is restricted to the even natural numbers. The generalisation to $J\neq 0$ is then trivial.

Rewriting (2.7) in the more appropriate form

$$\frac{k!^2(2l+2k+1)!}{(2k)!(l+k)!^2}S_l^{(1)}(k)=\frac{(k-1)!^2(2l+2k-1)!}{(2k-2)!(l+k-1)!^2}S_l^{(1)}(k-1)\qquad(\forall l\in\mathbb{N},\ k\in\mathbb{N}\ \&\ k\geqslant 1),\tag{2.8}$$

and defining a function $X_l(k)$ by:

$$X_l(k)=\left(\frac{\Gamma(k+1)}{\Gamma(k+l+1)}\right)^2\left(\frac{\Gamma(2k+2l+2)}{\Gamma(2k+1)}\right)S_l^{(1)}(k)\qquad(\forall l\in\mathbb{N},\ k\in\mathbb{C}),\tag{2.9}$$

the relation (2.8) can be written as:

$$X_l(k)=X_l(k-1)\qquad(\forall l\in\mathbb{N},\ k=1,2,\ldots).\tag{2.10}$$

It follows from (2.10) that

$$X_l(k)-X_l(0)=0\qquad(\forall l\in\mathbb{N},\ \forall k\in\mathbb{N}).\tag{2.11}$$

For any natural number l, the functions $X_l(k)$ and $X_l(k)-X_l(0)$ are rational, since by virtue of (2.9) and (1.4) they can be written for such l as a finite sum of ratios of polynomials. Since a rational function which has infinitely many zeros, vanishes identically, it is clear that $X_l(k)$ is constant with respect to the variable k.

In order to evaluate $X_l(k)$, we use the property $X_l(k)=X_l(\frac{1}{2})$. With the help of (2.9) and (1.4) one finds:

$$\begin{aligned}X_l(k)&=X_l(\tfrac{1}{2})\\&=\frac{\Gamma^2(\frac{3}{2})\Gamma(2l+3)}{\Gamma(2)\Gamma^2(l+\frac{3}{2})}\sum_{l'=0}^{l}\frac{1}{2(l'+1)}\begin{pmatrix}l & l' & l-l'\\0 & 0 & 0\end{pmatrix}^2\\&=\frac{(2l+2)!}{[(l+\frac{1}{2})(l-\frac{1}{2})\ldots\frac{3}{2}]^2}\frac{1}{2(l+1)}=\frac{(2l+1)!2^{2l}}{(2l+1)!!^2}=\frac{(2l+1)!}{l!^2}\left(\frac{(2l)!!}{(2l+1)!!}\right)^2.\end{aligned}\tag{2.12}$$

In the second step the result (2.3) has been used. Substituting (2.12) in (2.9), one obtains the following expression:

$$S_l^{(1)}(k)=\frac{\Gamma(2k+1)\Gamma^2(k+l+1)}{\Gamma^2(k+1)\Gamma(2k+2l+2)}\frac{(2l+1)!}{l!^2}\left(\frac{(2l)!!}{(2l+1)!!}\right)^2\qquad(\forall l\in\mathbb{N},\ \forall k\in\mathbb{C}).\tag{2.13}$$

Wherever it may occur, we set by convention (Abramowitz and Stegun 1970) $(0)!!=1$ and $(-1)!!=1$.

3. Special forms of $S_l^{(1)}(k)$

For integral and half-integral values of the variable k, (2.13) can be considerably simplified. The reader can easily verify that

$$S_l^{(1)}(t)=\frac{(2t)!(t+l)!^2}{t!^2(2t+2l+1)!}\frac{(2l+1)!}{l!^2}\left(\frac{(2l)!!}{(2l+1)!!}\right)^2\qquad(\forall t\in\mathbb{N}),\tag{3.1}$$

$$S_l^{(1)}(t+\tfrac{1}{2})=\frac{(2t+1)!(2l+1)!}{(2t+2l+2)!}\left(\frac{(2t+2l+1)!!}{(2t+1)!!(2l+1)!!}\right)^2\qquad(\forall t\in\mathbb{N}).\tag{3.2}$$

With a little algebra, the right-hand sides of (3.1) and (3.2) are brought into the more elegant forms:

$$S_l^{(1)}(t)=\frac{(2t-1)!!(2t+2l)!!(2l)!!}{(2t)!!(2t+2l+1)!!(2l+1)!!} \qquad (\forall t \in \mathbb{N}), \tag{3.3}$$

$$S_l^{(1)}(t+\tfrac{1}{2})=\frac{(2t)!!(2t+2l+1)!!(2l)!!}{(2t+1)!!(2t+2l+2)!!(2l+1)!!} \qquad (\forall t \in \mathbb{N}). \tag{3.4}$$

The functions $\Gamma(2k+1)$ and $\Gamma(k+1)$ have simple poles at the integer values of k satisfying $-l-1<k<0$. As $\Gamma(k+1)$ appears squared in the denominator of (2.13), it follows that:

$$S_l^{(1)}(-t)=0 \qquad (t \in \mathbb{N} \;\&\; 0<t<l+1). \tag{3.5}$$

We would like to remark that this result can also be deduced from equation (11) of Morgan (1976). A similar argument shows that

$$|S_l^{(1)}(-t-\tfrac{1}{2})|=\infty \qquad (t \in \mathbb{N} \;\&\; 0 \leqslant t<l+1). \tag{3.6}$$

This can also be seen directly from (1.4), since for these t-values there is always a l'-value making $S_l^{(1)}(k)$ infinite.

We next turn to the evaluation of $S_l^{(1)}(-t)$ for integers t satisfying $t \geqslant l+1$. One way to obtain an expression for these quantities is to calculate the residues of the respective functions $\Gamma(2k+1)$, $\Gamma(k+l+1)$, $\Gamma(k+1)$ and $\Gamma(2k+2l+2)$ at $k=-t$, with the help of the recursion formula:

$$\Gamma(x)=\frac{\Gamma(x+t+1)}{(x+t)(x+t-1)\ldots x}.$$

After a few calculations, one arrives at

$$S_l^{(1)}(-t)=-\frac{(t-1)!^2(2t-2l-2)!(2l+1)!}{(2t-1)!(t-l-1)!^2 l!^2}\left(\frac{(2l)!!}{(2l+1)!!}\right)^2 \qquad (t \in \mathbb{N} \;\&\; t \geqslant l+1). \tag{3.7}$$

An alternative method for finding $S_l^{(1)}(-t)$ is based on the property

$$S_l^{(1)}(-t)=\sum_{l'=0}^{l}\frac{1}{2l'-2t+1}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0\end{pmatrix}^2=\sum_{l'=0}^{l}\frac{1}{2l-2l'-2t+1}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0\end{pmatrix}^2,$$

or

$$S_l^{(1)}(-t)=-S_l^{(1)}(t-l-1). \tag{3.8}$$

The relation (3.8) is valid for all values of t. In particular it is found from (3.3) that

$$S_l^{(1)}(-t)=-\frac{(2t-2)!!(2t-2l-3)!!(2l)!!}{(2t-1)!!(2t-2l-2)!!(2l+1)!!} \qquad (t \in \mathbb{N} \;\&\; t \geqslant l+1) \tag{3.9}$$

which is an equivalent but more elegant form of (3.7). As it also follows from (3.8) that

$$S_l^{(1)}(-t-\tfrac{1}{2})=-S_l^{(1)}(t-l-\tfrac{1}{2})=-S_l^{(1)}(t-l-1+\tfrac{1}{2}), \tag{3.10}$$

one obtains by the aid of (3.4) a last simplified formula

$$S_l^{(1)}(-t-\tfrac{1}{2})=-\frac{(2t-1)!!(2t-2l-2)!!(2l)!!}{(2t)!!(2t-2l-1)!!(2l+1)!!} \qquad (t \in \mathbb{N} \;\&\; t \geqslant l+1). \tag{3.11}$$

Finally we notice that formulae (3.3), (3.4), (3.5), (3.6), (3.9) and (3.11) clearly demonstrate the property that

$$\frac{(2t)!!(2t+2l+1)!!}{(2t-1)!!(2t+2l)!!}S_l^{(1)}(t),$$

is independent of t ($2t \in \mathbb{Z}$) which is just a restricted version of (2.8), which was shown to be true even for complex k.

4. Evaluation of $S_l^{(2)}(k_1, k_2)$

The evaluation of $S_l^{(2)}(k_1, k_2)$ is particularly simple if $k_1 \neq k_2$. Indeed, taking into account the identity

$$\frac{1}{(2l'+2k_1+1)(2l'+2k_2+1)} = \frac{1}{2(k_2-k_1)}\left(\frac{1}{2l'+2k_1+1} - \frac{1}{2l'+2k_2+1}\right) \qquad (k_1 \neq k_2),$$

one immediately obtains

$$S_l^{(2)}(k_1, k_2) = \frac{1}{2(k_2-k_1)}(S_l^{(1)}(k_1) - S_l^{(1)}(k_2)) \qquad (k_1 \neq k_2). \tag{4.1}$$

Indeed Morgan (1976) has previously remarked that any summation of the form

$$\sum_{l'=0}^{l} f(l, z, l')\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2 \qquad (z \in \mathbb{C}),$$

can be evaluated with the help of the expressions derived for $S_l^{(1)}(k)$, provided that the function $f(l, z, l')$ can be written as a linear combination of reciprocals of terms linear in l'.

We now turn to the calculation of $S_l^{(2)}(k, k)$, further noted as $S_l^{(2)}(k)$. From the explicit form (1.4) of $S_l^{(2)}(k)$ it follows that

$$S_l^{(2)}(k) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}k}S_l^{(1)}(k). \tag{4.2}$$

Defining a function $\Phi(k)$ by

$$\Phi(k) = \frac{\Gamma(2k+1)\Gamma^2(k+l+1)}{\Gamma^2(k+1)\Gamma(2k+2l+2)}, \tag{4.3}$$

the essential point in the determination of $S_l^{(2)}(k)$ by equation (4.2), is the evaluation of $\mathrm{d}\Phi(k)/\mathrm{d}k \equiv \Phi'(k)$. One finds

$$\Phi'(k) = 2\Phi(k)(\psi(2k+1) + \psi(k+l+1) - \psi(k+1) - \psi(2k+2l+2)), \tag{4.4}$$

where $\psi(x)$ is the digamma function defined by

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x), \tag{4.5}$$

and with the properties (Abramowitz and Stegun 1970, p 258)

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{4.6}$$

$$\psi(nx)=\frac{1}{n}\sum_{p=0}^{n-1}\psi\left(x+\frac{p}{n}\right)+\ln n \qquad (n=2,3,\ldots). \tag{4.7}$$

Making use of (4.7), (4.4) is transformed to

$$\begin{aligned}\Phi'(k)=2\Phi(k)[&\tfrac{1}{2}(\psi(k+\tfrac{1}{2})+\psi(k+1))-\psi(k+1)+\psi(k+l+1)\\&-\tfrac{1}{2}(\psi(k+l+1)+\psi(k+l+\tfrac{3}{2}))]\\=2\Phi(k)[&-\tfrac{1}{2}(\psi(k+1)-\psi(k+\tfrac{1}{2}))-\tfrac{1}{2}(\psi(k+l+\tfrac{3}{2})-\psi(k+l+1))].\end{aligned} \tag{4.8}$$

In the literature on Γ and related functions, a function $\beta(x)$ is introduced (Gradshteyn and Ryzhik 1965, p 947)

$$\beta(x)=\frac{1}{2}\left(\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right)\right), \tag{4.9}$$

with the recursion property

$$\beta(x+1)=-\beta(x)+\frac{1}{x}. \tag{4.10}$$

On account of (4.9), (4.8) reduces to:

$$\Phi'(k)=-2\Phi(k)(\beta(2k+1)+\beta(2k+2l+2)). \tag{4.11}$$

From (2.13), (4.2) and (4.11) it follows that $S_l^{(2)}(k)$ can most generally be written as

$$S_l^{(2)}(k)=\frac{\Gamma(2k+1)\Gamma^2(k+l+1)}{\Gamma^2(k+1)\Gamma(2k+2l+2)}(\beta(2k+1)+\beta(2k+2l+2))\frac{(2l+1)!}{l!^2}\left(\frac{(2l)!!}{(2l+1)!!}\right)^2, \tag{4.12}$$

or in the more compact form

$$S_l^{(2)}(k)=S_l^{(1)}(k)(\beta(2k+1)+\beta(2k+2l+2)). \tag{4.13}$$

It has to be noted that the method used in § 2 for the evaluation of $S_l^{(1)}(k)$ is unsuccessful here, except for some special k-values. As an example, one can arrive by means of this method at the relation (for $l\neq 0$)

$$S_l^{(2)}(-1)\equiv\sum_{l'=0}^{l}\frac{1}{(2l'-1)^2}\begin{pmatrix}l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2=\frac{2l+1}{2l}\sum_{l'=0}^{l}\frac{1}{2l'+1}\begin{pmatrix}l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2 \tag{4.14}$$

which, due to (2.13), reduces to

$$S_l^{(2)}(-1)=\frac{(2l)!!(2l-2)!!}{(2l+1)!!(2l-1)!!}.$$

This result has also been mentioned by Morgan (1977). Up to now it was the only analytic expression available in the literature for quantities of the form (1.4) with n larger than 1.

The reason for the shortcoming of the method of § 2 for $n\geq 2$ and general k-values is that the functions $\psi(x)$ and $\beta(x)$ are additively recursive, as may be seen from (4.6), (4.10), whereas $\Gamma(x)$ satisfies a multiplicative recursion relation. Indeed, defining a function $Y_l(k)$ by

$$Y_l(k)=\frac{(2k)!!(2k+2l+1)!!}{(2k-1)!!(2k+2l)!!}S_l^{(2)}(k) \qquad (2k\in\mathbb{Z}), \tag{4.15}$$

one finds with the aid of (4.12), (4.10) and (3.12):

$$Y_l(k+\tfrac{1}{2})+Y_l(k)=\frac{(2l)!!}{(2l+1)!!}\left(\frac{1}{2k+1}+\frac{1}{2k+2l+2}\right) \qquad (2k\in\mathbb{Z}), \tag{4.16}$$

which in the present case is the relation to be compared to (2.10).

5. Special forms of $S_l^{(2)}(k)$

The function $\beta(x)$ has the following series representation (Gradshteyn and Ryzhik 1965, p 947):

$$\beta(x)=\sum_{p=0}^{\infty}\frac{(-1)^p}{x+p}, \tag{5.1}$$

and has simple poles at $x\in\mathbb{Z}_-$. With the aid of the well known result

$$\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p}=\ln 2, \tag{5.2}$$

one derives a finite sum representation of $\beta(n)$, $n\in\mathbb{N}$:

$$\beta(n)=(-1)^{n+1}\ln 2+\sum_{p=1}^{n-1}\frac{(-1)^{p+n+1}}{p}. \tag{5.3}$$

Using (5.3) in (4.13) twice one immediately finds from (3.3) and (3.4):

$$S_l^{(2)}(t)=\frac{(2t-1)!!(2t+2l)!!(2l)!!}{(2t)!!(2t+2l+1)!!(2l+1)!!}\sum_{p=2t+1}^{2t+2l+1}\frac{(-1)^{p+1}}{p} \qquad (\forall t\in\mathbb{N}), \tag{5.4}$$

$$S_l^{(2)}(t+\tfrac{1}{2})=\frac{(2t)!!(2t+2l+1)!!(2l)!!}{(2t+1)!!(2t+2l+2)!!(2l+1)!!}\sum_{p=2t+2}^{2t+2l+2}\frac{(-1)^{p}}{p} \qquad (\forall t\in\mathbb{N}). \tag{5.5}$$

By means of the property

$$S_l^{(2)}(-t)=S_l^{(2)}(t-l-1), \tag{5.6}$$

two other special forms of $S_l^{(2)}(k)$ are derived:

$$S_l^{(2)}(-t)=\frac{(2t-2)!!(2t-2l-3)!!(2l)!!}{(2t-1)!!(2t-2l-2)!!(2l+1)!!}\sum_{p=2t-2l-1}^{2t-1}\frac{(-1)^{p+1}}{p} \qquad (t\in\mathbb{N}\ \&\ t\geqslant l+1), \tag{5.7}$$

$$S_l^{(2)}(-t-\tfrac{1}{2})=\frac{(2t-1)!!(2t-2l-2)!!(2l)!!}{(2t)!!(2t-2l-1)!!(2l+1)!!}\sum_{p=2t-2l}^{2t}\frac{(-1)^{p}}{p} \qquad (t\in\mathbb{N}\ \&\ t\geqslant l+1). \tag{5.8}$$

We next turn to the evaluation of $S_l^{(2)}(k)$ for integers k satisfying $-1\geqslant k\geqslant -l$. This can be done by calculating the residues of the functions $\Gamma(2k+1)$, $\Gamma(k+1)$ and $\beta(2k+1)$, which have simple poles at these k-values. Another way to obtain $S_l^{(2)}(k)$ is to calculate $\Phi'(k)$ directly at $k=-t$ with $t<l+1$. Since $\Phi(-t)$ is zero due to the fact that the denominator has a double pole and the numerator only a single one, all functions which are regular and different from zero at $k=-t$, can be brought in front

of the differential operator. One thus finds:

$$\Phi'(k)\Big|_{k=-t}=\frac{\Gamma^2(l-t+1)}{\Gamma(2l-2t+2)}\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{\Gamma(2k+1)}{\Gamma^2(k+1)}\right)\Big|_{k=-t}$$

$$=\frac{(l-t)!^2}{(2l-2t+1)!}\left(\frac{\Gamma(2k+2t+1)(k+t)(k+t-1)\ldots(k+1)}{\Gamma(k+t+1)(2k+2t)(2k+2t-1)\ldots(2k+1)}\right)\Big|_{k=-t}$$

$$\times\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{\Gamma(k+1)}\right)\Big|_{k=-t}$$

$$=\frac{1}{2}\frac{(l-t)!^2(t-1)!}{(2l-2t+1)!(2t-1)!}(-1)^t\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{\Gamma(-k)}{\pi}\sin[\pi(k+1)]\right)\Big|_{k=-t}$$

$$=-\frac{1}{2}\frac{(l-t)!^2(t-1)!^2}{(2l-2t+1)!(2t-1)!},\tag{5.9}$$

where in one of the steps the property

$$\Gamma(x)\Gamma(1-x)=\pi\ \mathrm{cosec}\,(\pi x),$$

has been used. Substitution of (5.9) and (2.13) in (4.2) yields:

$$S_l^{(2)}(-t)=\frac{1}{4}\frac{(l-t)!^2(t-1)!^2}{(2l-2t+1)!(2t-1)!}\frac{(2l+1)!}{l!^2}\left(\frac{(2l)!!}{(2l+1)!!}\right)^2\qquad(t\in\mathbb{N}\ \&\ 0<t<l+1),$$

or after some simplifications

$$S_l^{(2)}(-t)=\frac{(2t-2)!!(2l-2t)!!(2l)!!}{(2t-1)!!(2l-2t+1)!!(2l+1)!!}\qquad(t\in\mathbb{N}\ \&\ 0<t<l+1).\tag{5.10}$$

For the sake of completeness we also note that

$$|S_l^{(2)}(-t-\tfrac{1}{2})|=\infty\qquad(t\in\mathbb{N}\ \&\ 0\leq t<l+1).\tag{5.11}$$

6. Evaluation of other $S_l^{(n)}(k_1,k_2,\ldots,k_n)$ functions

An expression for $S_l^{(0)}$ has already been given by Morgan (1977), and can also be derived with the technique outlined in § 2. One obtains:

$$S_l^{(0)}\equiv\sum_{l'=0}^{l}\begin{pmatrix}l & l' & l-l'\\ 0 & 0 & 0\end{pmatrix}^2=\frac{(2l)!!}{(2l+1)!!}.\tag{6.1}$$

The general case $S_l^{(n)}(k_1,k_2,\ldots,k_n)$ with $n>2$ can be treated by splitting the product $\Pi_{i=1}^n(2l'+2k_i+1)^{-1}$ in partial fractions. This shows that $S_l^{(n)}(k_1,k_2,\ldots,k_n)$ can be expressed as a sum of quantities $S_l^{(j)}(k_1,\ldots,k_j)$ with $j<n$, as long as not all k_i-values ($i=1,2,\ldots,n$) are equal. In the latter case one introduces the notation $S_l^{(n)}(k)=S_l^{(n)}(k,k,\ldots,k)$, and the generalisation of (4.2) reads:

$$S_l^{(n)}(k)=\frac{1}{(-2)^{n-1}(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}k^{n-1}}S_l^{(1)}(k).\tag{6.2}$$

By comparing (4.13) with (4.2) one finds that $S_l^{(1)}(k)$ satisfies the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}k} S_l^{(1)}(k) = -2S_l^{(1)}(k)(\beta(2k+1)+\beta(2k+2l+2)),$$

showing us that $S_l^{(1)}(k)$ can also be written in the form

$$S_l^{(1)}(k) = C\,\mathrm{e}^{R(k)}, \tag{6.3}$$

with

$$R'(k) = -2(\beta(2k+1)+\beta(2k+2l+2)). \tag{6.4}$$

By means of Faà di Bruno's differentiation formula (Abramowitz and Stegun 1970, p 823) one then obtains from (6.3):

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}k^{n-1}} S_l^{(1)}(k) = S_l^{(1)}(k) \sum_a (n-1, a_1, \ldots, a_{n-1})'(R'(k))^{a_1} \ldots (R^{(n-1)}(k))^{a_{n-1}}, \tag{6.5}$$

where the sum is taken over all natural numbers, $a_1, a_2, \ldots, a_{n-1}$, satisfying

$$a_1 + 2a_2 + \ldots + (n-1)a_{n-1} = n-1, \tag{6.6}$$

$$a_1 + a_2 + \ldots + a_{n-1} \leqslant n-1, \tag{6.7}$$

and where $(n-1, a_1, \ldots, a_{n-1})'$ is given by

$$(n-1, a_1, \ldots, a_{n-1})' = (n-1)!/(1!)^{a_1} a_1!(2!)^{a_2} a_2! \ldots [(n-1)!]^{a_{n-1}} a_{n-1}!. \tag{6.8}$$

Consequently, $S_l^{(n)}(k)$ can be written as

$$S_l^{(n)}(k) = S_l^{(1)}(k) \sum_a \left(\prod_{i=1}^{n-1} \frac{-1^{n+a_i-1}}{(i!)^{a_i} a_i!} (\beta^{(i-1)}(2k+1) + \beta^{(i-1)}(2k+2l+2))^{a_i} \right), \tag{6.9}$$

where the summation convention (6.6), (6.7) is understood.

In the right-hand side of (6.9) the sum of the β functions can be written as a finite sum for all non-negative integral and half-integral values of k. Indeed, $\beta^{(n)}(x)$ has the series representation (Gradshteyn and Ryzhik 1965, p 947)

$$\beta^{(n)}(x) = (-1)^n n! \sum_{p=0}^{\infty} \frac{(-1)^p}{(x+p)^{n+1}}, \tag{6.10}$$

whereas

$$\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p^n} = (1-2^{1-n})\zeta(n) \qquad (\mathrm{Re}\, n > 0), \tag{6.11}$$

$\zeta(n)$ being the Riemann zeta function. From (6.10) and (6.11) it follows that

$$\beta^{(i-1)}(2t+1) + \beta^{(i-1)}(2t+2l+2) = (-1)^{i+2t}(i-1)! \sum_{p=2t+1}^{2t+2l+1} \frac{(-1)^p}{p^i} \qquad (2t \in \mathbb{N}). \tag{6.12}$$

With the relation

$$S_l^{(n)}(-t) = (-1)^n S_l^{(n)}(t-l-1), \tag{6.13}$$

the result can then be continued to all integral and half-integral values of k satisfying $k \leqslant -l-1$, while for integral values of k satisfying $-l-1 < k < 0$ a residue calculation is needed for the evaluation of $S_l^{(n)}(k)$.

7. Conclusions

The sums of the form

$$\sum_{l'=0}^{l}\left(\prod_{i=1}^{n}\frac{1}{2l'+2k_i+1}\right)\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2$$

are analytically expressed in terms of gamma functions and higher transcendentals. It has been shown that the general result deduced contains the relations, previously proved by Morgan (1975, 1976, 1977), Rashid (1976) and Vanden Berghe and De Meyer (1976) as particular cases. A set of simple orthogonality relations of the form

$$S_l^{(1)}(-t)\equiv\sum_{l'=0}^{l}\frac{1}{2l'-2t+1}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2=0 \qquad (t\in\mathbb{N}\ \&\ 0<t<l+1),$$

has been obtained. They represent a generalisation of a previously proved orthogonality relation (Morgan 1975), which corresponds to the $t=1$ case. For $n\geqslant 2$ special attention has been drawn to those of the sums considered, where two or more of the k_i-values coincide. Although Morgan (1977) alleged that the evaluation of this kind of summation is difficult for general k_i and n, we were able to derive in the last section a complete general analytical expression for relations of the form

$$S_l^{(n)}(k)=\sum_{l'=0}^{l}\frac{1}{(2l'+2k+1)^n}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2.$$

Acknowledgments

We would like to express our gratitude to Professor Dr C C Grosjean for his interest during the course of this work.

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